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A NOTE ON MONOTONE CONVERGENCE TO SOLUTIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

By

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P-1012 DH

January 16, 1957

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SUMMARY

The purpose of this brief note is to show that we can obtain a monotone increasing sequence of approximations to the solution of the differential equation

$$(1) \qquad \frac{du}{dt} = \phi(u,t), \; u(o) = c,$$

provided that we assume that $\phi(u,t)$ is a twice differentiable convex function of u in some t-interval $[o,t_o]$. Similarly, monotone decreasing sequences can be obtained if ϕ is concave.

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§1. Introduction.

The purpose of this brief note is to show that we can obtain a monotone increasing sequence of approximations to the solution of the differential equation.

(1) $$\frac{du}{dt} = \phi(u,t), \quad u(o) = c,$$

provided that we assume that $\phi(u,t)$ is a twice differentiable convex function of u in some t-interval $[o,t_o]$. Similarly, monotone decreasing sequences can be obtained if ϕ is concave.

The connection between the methods and results discussed here and the theory of dynamic programming is treated in [1].

§2. The Riccati Equation.

Let us begin by considering a particularly important case, the Riccati equation,

(1) $$\frac{du}{dt} = u^2 + a(t), \quad u(o) = c.$$

Assume that a(t) is integrable over some initial interval $o \leq t \leq t_o$.

The basic relation we shall employ is

(2) $$u^2 = \max_v \; (2uv - v^2).$$

We can then write (1) in the form

(3) $$\frac{du}{dt} = \underset{v}{\mathrm{Max}} \left[2uv - v^2 + a(t) \right], \quad u(o) = c,$$

where v ranges over all functions defined over $[o, t_o]$.

As pointed out in [1], (3) leads to the result

(4) $$u = \underset{v}{\mathrm{Max}}\ \mho(v,t),$$

where $\mho$ is the solution of

(5) $$\frac{d\mho}{dt} = 2\mho v - v^2 + a(t), \quad \mho(o) = c,$$

for a fixed function $v = v(t)$.

Let us now use the representation in (3) to obtain a monotone increasing sequence of approximations to the solution of (1). Let $v_o(t)$ be an integrable function over $[o, t_o]$, and let u_o be defined as the solution of

(6) $$\frac{du_o}{dt} = 2\, u_o v_o - v_o^{\,2} + a(t), \quad u_o(o) = c.$$

Let $v_1(t)$ be the function which maximizes the quantity

(7) $$g(v) = 2\, u_o v - v^2 + a(t),$$

which is to say $v_1 = u_o$. Now define the next approximation u_1 as the solution of

$$\frac{du_1}{dt} = 2\, u_1 v_1 - v_1^{\,2} + a(t), \quad u_1(o) = c.$$

In this way we obtain the recurrence relations

(8) $$\frac{du_{n+1}}{dt} = 2\, u_{n+1}\, u_n - u_n^{\,2} + a(t), \quad u_{n+1}(o) = c,$$

$n=0,1,2,\ldots$, which we recognize as Newton's method applied to the original equation in (1).

Let us note in passing that the method we have employed to derive this sequence of approximations is a particular application of the concept of "approximation in policy space", cf. [2].

We wish to show that

(9) $$u_o \leq u_1 \leq \cdots \leq u_n \leq \cdots,$$

for $o \leq t \leq t_1$, where $[o, t_1]$ is some common interval of definition.

It is easy to show using standard techniques that we can ensure a common interval. Let us consequently turn to the proof of monotonicity. We have

(10) $$\frac{du_n}{dt} = 2\, u_n u_{n-1} - u_{n-1}^2 + a(t) \leq 2\, u_n u_n - u_n^2 + a(t),$$

since $v = u_n$ maximizes $2\, u_n v - v^2 + a(t)$. Comparing this last equation with (8), we see that we will have established the desired monotonicity if we can prove

Lemma: *Let $x(t)$ be a function satisfying the inequality*

(11) $$\frac{dx}{dt} \leq p(t)x + q(t),\ x(o) = c,\ t \geq o,$$

and $y(t)$ satisfy the equality

(12) $$\frac{dy}{dt} = p(t)y + q(t),\ y(o) = c,\ t \geq o,$$

then

(13) $$x(t) \leq y(t),$$

for $t \geq o$.

Proof: The relation in (11) is equivalent to

$$\frac{dx}{dt} = p(t)x + q(t) - r(t), \quad x(o) = c, \tag{14}$$

with $r(t) \geq o$. Solving for x we see that

$$x = y - \int_o^t r(s) e^{\int_s^t p(z)dt} ds, \tag{15}$$

which demonstrates (13).

§3. The General One-Dimensional Case.

Consider the equation

$$\frac{du}{dt} = \phi(u,t), \quad u(o) = c. \tag{1}$$

Let us now prove

Theorem 1. *The sequence of Approximations* $\{u_m\}$ *defined by*

$$\frac{du_o}{dt} = \phi(v_o,t) + (u_o - v_o)\, \phi'(v_o,t), \quad u_o(o) = c, \tag{2}$$

$$\frac{du_{n+1}}{dt} = \phi(u_n,t) + (u_{n+1} - u_n)\, \phi'(u_n,t), \quad u_{n+1}(o) = c,$$

(' denotes the partial derivative with respect to u)

is monotone increasing, $u_o \leq u_1 \leq \cdots \leq u_n \leq \cdots$, *within a common interval* $[o, t_o]$, *provided that* $v_o(t)$ *is an integrable function over some interval* $[o, t_1]$, *and that the function* $\phi(u,t)$ *possesses a non-negative second derivative in u for* $o \leq t \leq t_1$.

Proof: The first point to observe is that the generalization of (2.2) is

$$\phi(u,t) = \operatorname*{Max}_v \left[\phi(v,t) + (u-v)\, \phi'(v,t) \right]. \tag{3}$$

The existence of a common interval of existence follows standard lines, and the remainder of the proof follows the lines of §2.

§4. Multi-Dimensional Case.

If $f(x,y,t)$ and $g(x,y,t)$ are convex functions of x and y within some t-interval, it is easy to see that we may write the system of equations

$$(1)\qquad \frac{dx}{dt} = f(x,y,t),\ x(o) = a,$$
$$\frac{dy}{dt} = g(x,y,t),\ y(o) = b,$$

in the form

$$(2)\qquad \frac{dx}{dt} = \underset{u,v}{\text{Max}} \left[f(u,v,t) + (x-u)\frac{\partial f}{\partial u} + (y-v)\frac{\partial f}{\partial v} \right],\ x(o) = a,$$

$$\frac{dy}{dt} = \underset{u,v}{\text{Max}} \left[g(u,v,t) + (x-u)\frac{\partial g}{\partial u} + (y-v)\frac{\partial g}{\partial v} \right],\ y(o) = b,$$

The analogue of the Lemma in §2 is not unrestrictedly true, however. If x and y satisfy the inequalities

$$(3)\qquad \frac{dx}{dt} \le a_{11}(t)x + a_{12}(t)y + r_1(t),\ x(o) = c,$$
$$\frac{dy}{dt} \le a_{21}(t)x + a_{22}(t)y + r_2(t),\ y(o) = c,$$

it is not always true that x and y are respectively bounded by the solutions of equalities.

For the case where the a_{ij} are constants, a necessary and sufficient condition that this be so is that $a_{ij} \ge o$, $i \ne j$, and for the general case a simple sufficient condition is $a_{ij}(t) \ge o$, with a necessary condition of quite complicated and unusable type, cf. [3], p. 14.

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